NON-STANDARD METHOD FOR SOLVING HYPERBOLIC HEAT EQUATIONS

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Abstract. In this article, we build upon the pioneering work of Abraham Temkin (1919-2007), who introduced a novel separation of variables method for non-stationary heat conduction in the 1960s. Our extension applies this method to the hyperbolic heat equation, incorporating a relaxation term. The hyperbolic heat equation, a partial differential equation combining features of both hyperbolic and parabolic equations, finds wide applications across various scientific fields, including physics, engineering, geophysics, medical imaging, and more. Our investigation centres on the application of the Temkin's method to the hyperbolic heat equation, with the aim to provide insights into its effectiveness in solving direct and inverse problems. The method relies on the observation that for non-stationary heat conduction with dynamic boundary conditions, the influence of initial conditions on the temperature distribution diminishes over time. Consequently, it is reasonable to assume that the temperature distribution is primarily influenced by the time-dependent boundary conditions. By expressing the solution to the given problem as a series, where each term is a product of a derivative of the given boundary condition and an unknown function P dependent on a spatial variable, we obtain a set of ordinary differential equations. These equations lead us to deduce the spatial functions, which are found to be polynomial in nature. While this approach holds promise for formulating an inverse problem to determine the speed of propagation, our current numerical results are inconclusive.

Keywords: hyperbolic equation, non-standard method, direct problem, inverse problem.

Introduction

Both parabolic and hyperbolic heat equations are fundamental models for heat transfer phenomena, each offering distinct properties and applications. While the parabolic heat equation provides a standard framework for many heat conduction problems, the hyperbolic heat equation, often referred to as the relativistic hyperbolic equation, offers notable advantages. Unlike the parabolic equation, the hyperbolic equation accounts for the finite propagation speed of heat waves, making it particularly suitable for scenarios characterized by rapid temperature changes and wave-like behaviour. Consequently, understanding hyperbolic heat conduction is crucial for various engineering and scientific applications, with research focusing on direct and inverse analysis methods.

Significant contributions to the field have been made by researchers worldwide, including works like [1-3]. In the latter of these studies, for example, various heat equations including the parabolic heat equation based on the Fourier's theory, the hyperbolic heat equation, and the relativistic heat equation are solved analytically for modelling thermal ablation of biological tissues and the temperature distributions compared. Additionally, inverse problems for the hyperbolic heat equation have been explored in studies such as [5]. Moreover, implementations of hyperbolic thermal conduction in smoothed particle hydrodynamics, a computational method primarily used in fluid dynamics, have been investigated, e.g. in [4].

Our paper introduces the Temkin's method, a novel separation of variables technique tailored for solving heat conduction equations in one-dimensional finite domains. While traditional approaches like finite difference schemes have limitations, the Temkin's method offers a promising alternative.

Mathematical model of direct problem

We consider a one-dimensional problem of heat propagation over a finite domain bounded by $x = 0$ and $x = l$, where l denotes the length of the segment, and the temperature field $u(x,t)$ in this case is governed by

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\tau}\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, x \in (0,l), t > 0 \tag{1}$$

Here $c = \sqrt{\frac{k}{\tau}}$ is its characteristic speed, k is the thermal diffusivity and τ is a relaxation time which depends on the mechanism of heat transport.

Let us assume that the behaviour of the function $u(x,t)$ is governed by the following conditions – initial conditions, representing the initial temperature distribution and temperature flux within the interval at $t = 0$:

$$u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = 0, x \in (0,l) \tag{2}$$

and boundary conditions, imposing the temperature at the x -boundaries of the interval:

$$u(0,t) = 0, u(l,t) = u_1(t) \tag{3}$$

As per [6], the analytical solution of the problem (1)-(3) is a function that is defined by this formula

$$u(x,t) = -c^2 \cdot \int_0^t u_1(\iota)\left[\frac{\partial G(x,\xi,t-\iota)}{\partial \xi}\right]_{\xi = l} d\iota \tag{4}$$

Assuming that $c^2\pi^2n^2 - \frac{l^2}{4\tau^2} \leq 0$ for $n = \overline{1,m}$ and $c^2\pi^2n^2 - \frac{l^2}{4\tau^2} > 0$ for $n = \overline{m+1,\infty}$, the so-called Green's function for this hyperbolic heat equation problem is defined by

$$\begin{aligned} G(x,\xi,t) = \frac{2}{l}e^{-\frac{t}{2\tau}} \cdot \sum_{n=1}^{m} \sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n \xi}{l}\right)\frac{\sinh\left(t\sqrt{\beta_n}\right)}{\sqrt{\beta_n}} + \\ + \frac{2}{l}e^{-\frac{t}{2\tau}} \cdot \sum_{n=m+1}^{\infty} \sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n \xi}{l}\right)\frac{\sin\left(t\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} + \end{aligned} \tag{5}$$

where $\beta_n = \frac{1}{4\tau^2} - \frac{c^2\pi^2n^2}{l^2}$ and $\lambda_n = \frac{c^2\pi^2n^2}{l^2} - \frac{1}{4\tau^2}$.

There are several techniques available for solving partial differential equations, with the Green's function method being just one example. Another commonly employed approach is the method of separation of variables. The primary objective of this method is to simplify the solution process for partial differential equations by decomposing them into simpler ordinary differential equations. This is achieved by expressing the solution as a sum or product of functions, each depending on only one independent variable. The separation of variables method, originally proposed by A.Temkin [7] and utilized in many works (e.g. in [8-10]), is applied herein to address the given problem (1)-(3). This method assumes that the solution takes the form:

$$u(x,t) = \sum_{n=0}^{\infty} P_n(x)T_n(t) \tag{6}$$

where $T_n(t)$ refers to the n^{th} derivative of the boundary condition (3) with respect to x evaluated at $x = l$:

$$T_n(t) = \frac{\partial^n u}{\partial t^n}(l,t) \text{ and } T_0 = u_1(t), \tag{7}$$

but functions $P_n(x)$ depend on x .

Substitution of (6) into the hyperbolic heat equation (1) gives:

$$\begin{aligned} P_0(x)T_0''(t) + \frac{1}{\tau}P_0(x)T_0'(t) + P_1(x)T_1''(t) + \frac{1}{\tau}P_1(x)T_1'(t) + \\ P_2(x)T_2''(t) + \frac{1}{\tau}P_2(x)T_2'(t) + P_3(x)T_3''(t) + \frac{1}{\tau}P_3(x)T_3'(t) + \ldots = \\ = c^2(P_0''(x)T_0(t) + P_1''(x)T_1(t) + P_2''(x)T_2(t) + P_3''(x)T_3(t) + \ldots), \end{aligned} \tag{8}$$

where primes represent differentiation with respect to the appropriate argument. Considering that $T_i'(t) = T_{i+1}(t)$ (from (7)), equation (8) yields a system of these ordinary differential equations for $P_n(x)$:

$$P_0''(x) = 0$$
$$\frac{1}{\tau}P_0(x) = c^2 P_1''(x) \tag{9}$$
$$P_{i-1}(x) + \frac{1}{\tau}P_i(x) = c^2 P_{i+1}''(x) \text{ for } i = \overline{1,\infty}$$

Inserting (6) into the boundary conditions (3), we find:

$$P_0(0)T_0(t) + P_1(0)T_1(t) + P_2(0)T_2(t) + P_3(0)T_3(t) + \ldots = 0$$

and

$$P_0(l)T_0(t) + P_1(l)T_1(t) + P_2(l)T_2(t) + P_3(l)T_3(t) + \ldots = u_1(t)$$

which gives

$$P_i(0) = 0 \text{ for } i = \overline{0,\infty} \tag{10}$$
$$P_0(l) = 1, P_i(l) = 0 \text{ for } i = \overline{1,\infty} \tag{11}$$

Solving the system of equations (9) applying the boundary conditions (10)-(11) results into

$$P_0(x) = \frac{x}{l}$$
$$P_1(x) = \frac{1}{c^2\tau}\left(\frac{x^3}{6l} - \frac{lx}{6}\right) \tag{12}$$
$$P_2(x) = \frac{1}{c^2}\left(\frac{x^3}{6l} - \frac{lx}{6} + \frac{1}{c^2\tau^2}\left(\frac{x^5}{120l} - \frac{lx^3}{36} + \frac{7l^3x}{360}\right)\right)$$

etc.

Hence, we obtain a solution to the original problem (1)-(3) in the form described by equation (6), with spatial functions $P_n(x)$ given by equations (12).

Mathematical model of inverse problem

In a manner akin to our approach detailed in paper [10], we frame the inverse problem as a coefficient inverse problem where we want to determine the coefficient c . One way to go around this is to use the solution provided by the Temkin's method. In this approach, we approximate the infinite series by means of a finite series containing the first M terms. If we have knowledge of the temperature at an inner point ξ^* within the interval $(0, l)$, the approximation becomes

$$u(\xi^*, t) = \sum_{n=0}^{M} P_n(\xi^*)T_n(t) \tag{13}$$

If $M \geq 2$, the expression (13) becomes nonlinear with respect to $\frac{1}{c^2}$. To maintain the linearity, let us consider only the first two terms of the series, and the equation (13) becomes

$$u(\xi^*, t) = \frac{\xi^*}{l}u_1(t) + \frac{1}{c^2\tau}\left(\frac{\xi^{*3}}{6l} - \frac{\xi^* l}{6}\right)$$

When solving this for c^2, we get

$$c^2 = \frac{\left(\frac{\xi^{*3}}{6l} - \frac{\xi^* l}{6}\right)u_1''(t)}{\tau^2\left(u(\xi^*, t) - \frac{\xi^*}{l}u_1(t)\right)} \tag{14}$$

Results and discussion

We have conducted numerical experiments to solve the direct problem (1) - (3) with boundary conditions using solutions (4), (5) and (6), (12), as well as for the inverse problem. For these experiments, we selected specific parameter values $k = 10^{-4}$, $l = 0.02\,m$, $\omega = 0.02$, $\tau = 10^{-6}$, $A = 2$, set the initial condition to be

$$u_1 = A\sin\left(\omega t + \frac{3\pi}{2}\right) + A$$

and investigated the problem over the time interval $[0,320]$.

In Fig. 1 we have functions $P_n(x)$. These graphs suggest that the functions tend towards zero as n increases.

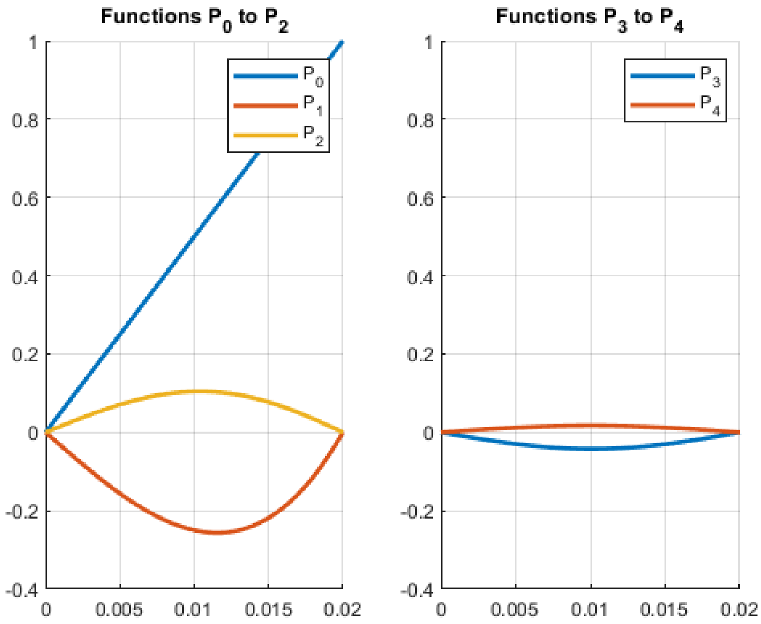

Fig. 1. **Functions P_n**

As for the function $u(x,t)$, we have the graph in Fig. 2 for the solution (4), (5), when taking the four terms of the series, and Fig. 3 for the solution found using the Temkin's method when taking the first five terms of the series.

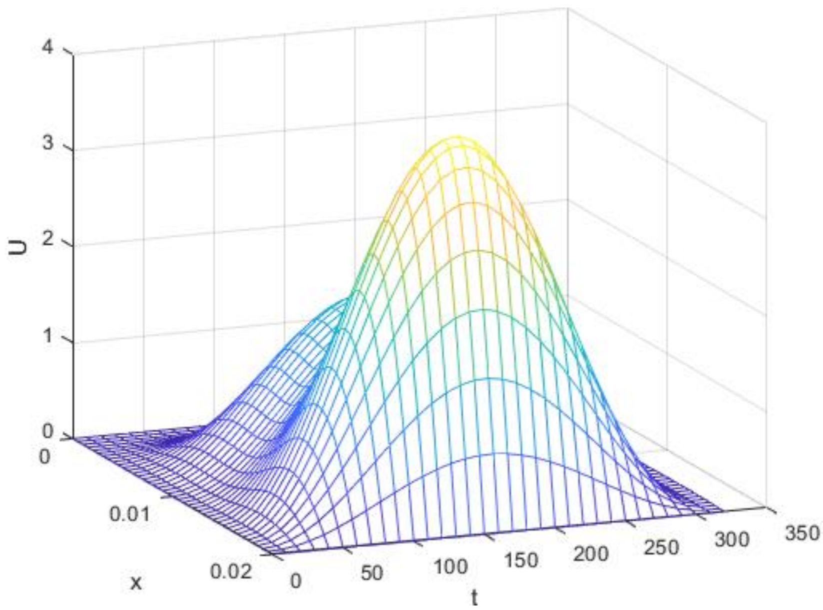

Fig. 2. **Exact solution using the Green's function method**

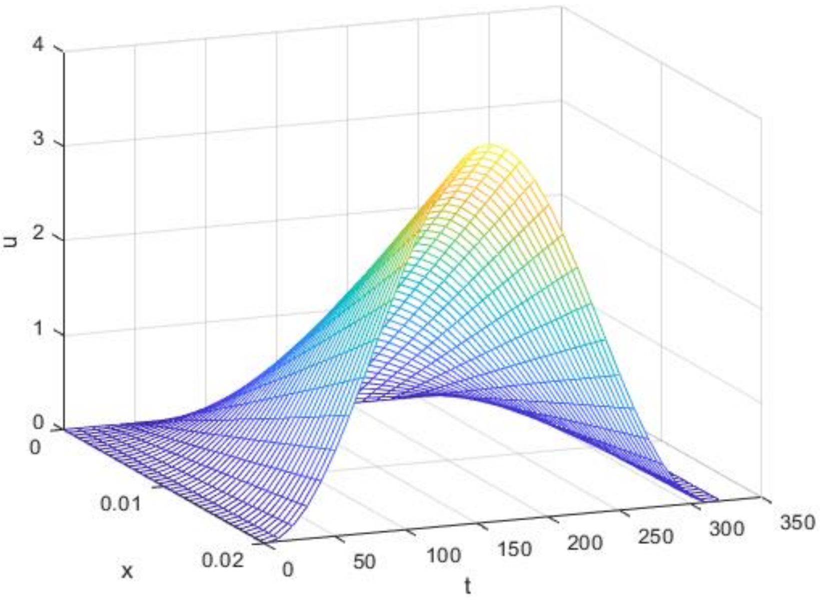

Fig. 3. **Exact solution using the Temkin's method**

Conclusions

1. The application of the Temkin's method to hyperbolic heat equations may be viable under certain conditions, particularly when considering a greater number of terms in the series.
2. One notable advantage of the Temkin's method is its relative simplicity compared to alternative approaches for solving direct hyperbolic equations. Our findings indicate that when applying the Temkin's method to the direct problem of the hyperbolic heat equation, it leads to quicker results compared to the traditional Green's function method.
3. While this study has focused primarily on solving the direct problem of the hyperbolic heat conduction equation using the Temkin's method, it is important to note that the second and third boundary conditions were not explicitly considered in this analysis.
4. Further investigation is required for the inverse coefficient problem, as the current numerical results are inconclusive.

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Author contributions

Conceptualization, Iltins I; software, formal analysis, investigation and writing, Treilande T. All authors have read and agreed to the published version of the manuscript.

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